

THE RICCI FLOW APPROACH TO HOMOGENEOUS EINSTEIN METRICS ON FLAG MANIFOLDS

STAVROS ANASTASSIOU AND IOANNIS CHRYSIKOS

ABSTRACT. We give the global picture of the normalized Ricci flow on generalized flag manifolds with two or three isotropy summands. The normalized Ricci flow for these spaces descends to a parameter depending system of two or three ordinary differential equations, respectively. We present here the qualitative study of these system's global phase portrait, by using techniques of Dynamical Systems theory. This study allows us to draw conclusions about the existence and the analytical form of invariant Einstein metrics on such manifolds, and seems to offer a better insight to the classification problem of invariant Einstein metrics on compact homogeneous spaces.

2000 *Mathematics Subject Classification.* Primary 53C25, 57M50, Secondary 37C10.

Keywords: Normalized Ricci flow, homogeneous Einstein metrics, flag manifolds, Poincaré compactification

INTRODUCTION

The Ricci flow equation was introduced by Richard Hamilton ([18]) in 1982, in order to study Thurston's geometrization conjecture on 3-manifolds. It is defined as:

$$\frac{\partial g}{\partial t} = -2 \operatorname{Ric}_g \quad (1)$$

with initial condition $g(0) = g_0$. Here $g = g(t)$ is a curve on the space of Riemannian metrics \mathcal{M} on a smooth manifold M^n , and Ric_g is the Ricci tensor of the Riemannian metric g . Ricci flow is a way to deform a Riemannian manifold (M, g) under a nonlinear evolution equation in order to process, or improve it. In a more general context, Ricci flow is a very useful tool in proving topological theorems in Riemannian and Kählerian manifolds. The most astonishing application of it is the proof of the Poincaré conjecture, due to Grisha Perelman's developments on the Hamilton's program.

Note that equation (1) does not preserve volume in general. When the underlying manifold M^n is compact one often considers the *normalized Ricci flow*

$$\frac{\partial g}{\partial t} = -2 \operatorname{Ric}_g + \frac{2r}{n} g, \quad (2)$$

where $r = r(g(t)) = \int_M S_g du_g / \int_M du_g$, du_g is the volume element of g , and S_g denotes the scalar curvature function of g , i.e. the trace of the Ricci tensor Ric_g . Under this normalized flow, the volume of the solution metric is constant in time. Equations (1) and (2) are equivalent by reparametrizing time t and scaling the metric in space by a function of t .

Fixed points of the normalized Ricci flow (2) are precisely the Einstein metrics, i.e. Riemannian metrics of constant Ricci curvature, that is $\operatorname{Ric}_g = c \cdot g$ for some constant $c \in \mathbb{R}$. The last equation reduces to a system of second order PDEs and general existence results are difficult to obtain. On compact manifolds M^n ($n \geq 3$), Einstein metrics of volume 1 are in a natural way privileged metrics, since they arise as the critical points of the total scalar functional $\mathbf{S}(g) = \int_M S_g du_g$, restricted to the space \mathcal{M}_1 of all Riemannian metrics of volume 1 (cf. [7], [23]).

One of the main problems about the Ricci flow is to determine the evolution process of the metrics defined on a manifold. In general, one starts with a metric g_0 on M that satisfies some rather general curvature condition \mathcal{R} and proves that as the normalized Ricci flow runs, the metric g_t converges to a limiting metric (as $t \rightarrow \infty$) which satisfies a more restrictive curvature condition \mathcal{R}' (cf. [18],[19]).

Recall that a Riemannian manifold (M, g) is called G -homogeneous if there is a closed subgroup G of the group of isometries $\text{Iso}(M, g)$ which acts transitively on M , that is, for any $p, q \in M$ there exists $g \in G$ with $gp = q$. Then $M = G/K$, where $K = \{g \in G : gp = p\}$ is the isotropy group at the point $p \in M$. Since G is a closed subgroup of $\text{Iso}(M, g)$, the isotropy subgroup K is a compact subgroup of $\text{Iso}(M, g)$, and thus M is compact, if and only if G is compact. When $M = G/K$ is a homogeneous space it is convenient to work with G -invariant Riemannian metrics $g = g(t)$, i.e. metrics for which the translations $\tau_a : G/K \rightarrow G/K$, $gK \mapsto agK$ act as isometries. For such metrics the non-linear system of PDEs which forms the Ricci flow equation, reduces to a non-linear system of ODEs. For this reason one can proceed to a study of the global behaviour of the Ricci flow, using tools from the theory of dynamical systems (cf. [8], [14] and [16]).

Let G be a compact, connected and simple Lie group. In this paper we study the global behaviour of the normalized Ricci flow, from a qualitative point of view, for a G -invariant Riemannian metric on generalized flag manifolds $M = G/K$, whose isotropy representation decomposes into two or three pairwise inequivalent irreducible K -modules (see Tables 1 and 3). Recall that a flag manifold is an adjoint orbit of G and any such space can be expressed as a homogeneous space of the form $G/C(S)$, where $C(S)$ is the centralizer of a torus $S \subset G$. Flag manifolds exhaust the compact and simply connected homogeneous Kähler manifolds and thus they have important applications in the physics of elementary particles, where they give rise to a broad class of supersymmetric sigma models. Such a homogeneous space admits a finite number of G -invariant complex structures and the first Chern class is positive. Also, for any complex structure there is a (unique) G -invariant Kähler-Einstein metric (cf. [2], [3]).

The explicit form of the normalized Ricci flow is given here and its qualitative properties are presented. We determine the asymptotical limit for a G -invariant Riemannian metric of $M = G/K$ and use the results to classify all homogeneous Einstein metrics on M . These metrics correspond to the singularities of the normalized Ricci flow located at infinity. In particular, by using the compactification method of Poincaré ([15]), we are able to obtain the fixed points coordinates and determine explicitly the coefficients of the Einstein metrics. Thus we prove the following theorems:

Theorem 1. *Let $M = G/K$ be a generalized flag manifold with two isotropy summands. The normalized Ricci flow on the space of G -invariant Riemannian metrics on M , possesses no finite singularities and exactly two at infinity. One of them is a repelling node and the second one is an attractive node. These fixed points determine explicitly the two (up to scale) invariant Einstein metrics of M .*

Theorem 2. *Let $M = G/K$ be a generalized flag manifold of a compact simple Lie group G with three isotropy summands, which is defined by painting black only one simple root in the Dynkin diagram of G . The normalized Ricci flow on the space of invariant Riemannian metrics on M , possesses no finite singularities and exactly three at infinity. One of them is a repelling node while the other two are saddle points. These fixed points determine explicitly the three (up to scale) G -invariant Einstein metrics of M .*

The paper is structured as follows: In Section 1 we recall some basic facts of the geometry of a generalized flag manifold $M = G/K$ and we describe the form of the normalized Ricci flow for

a G -invariant metric. In Section 2 we study flag manifolds with two or three isotropy summands and we give explicitly the system which determines this flow. In the next section we describe the Poincaré compactification method in the two dimensional case, and give the necessary formulas for the calculations that follow. The final section contains the global structure of the normalized Ricci flow and the proofs of Theorems 1 and 2.

1. THE NORMALIZED RICCI FLOW ON GENERALIZED FLAG MANIFOLDS

1.1. Generalized flag manifolds. A generalized flag manifold is an adjoint orbit of a compact semisimple Lie group G . Recall that G acts on its Lie algebra $\mathfrak{g} = T_e G$ through the adjoint action, i.e. the action induced by the adjoint representation $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ of G . Given an element $w \in \mathfrak{g}$ (i.e., a left-invariant vector field), the adjoint orbit of w is given by $M = \text{Ad}(G)w = \{\text{Ad}(G)w : g \in G\} \subset \mathfrak{g}$. Thus M is an imbedded manifold in an Euclidean space, the Lie algebra of G . Let $K = \{g \in G : \text{Ad}(g)w = w\} \subset G$ be the isotropy subgroup of w and let $\mathfrak{k} = T_e K$ be the corresponding Lie algebra. Since G acts on M transitively, M is diffeomorphic to the (compact) homogeneous space G/K , that is $\text{Ad}(G)w = G/K$. Then one can prove that the Lie algebra \mathfrak{k} is given by $\mathfrak{k} = \{X \in \mathfrak{g} : [X, w] = 0\} = \ker \text{ad}(w)$, where $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ is the adjoint representation of \mathfrak{g} . Moreover, the set $S_w = \{\exp(tw) : t \in \mathbb{R}\}$ is a torus in G and the isotropy subgroup K is identified with the centralizer of S_w in G , i.e. $K = C(S_w)$ (cf. [7]). From this fact it follows that $\text{rk } G = \text{rk } K$ and that K is connected.

Definition 1. *Let G be a compact semisimple Lie group. A generalized flag manifold is the adjoint orbit of an element in the Lie algebra \mathfrak{g} of G . Equivalently, it is a homogeneous space of the form G/K , where $K = C(S) = \{g \in G : ghg^{-1} = h \text{ for all } h \in S\}$ is the centralizer of a torus S in G .*

1.2. Geometry of flag manifolds. In order to construct the normalized Ricci flow on a generalized flag manifold it is convenient to view it as a homogeneous space and not as an imbedded submanifold of an Euclidean space.

Let $M = G/K = G/C(S)$ be a generalized flag manifold. Since G is compact and semisimple, the Killing form B of $\mathfrak{g} = T_e G$ is non-degenerate and negative definite. Thus the bilinear form $(\cdot, \cdot) = -B(\cdot, \cdot)$ is an $\text{Ad}(G)$ -invariant inner product on \mathfrak{g} . Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a reductive decomposition of \mathfrak{g} with respect to (\cdot, \cdot) , that is $\text{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$. Then, as usual, we can identify the $\text{Ad}(K)$ -invariant subspace \mathfrak{m} with the tangent space $T_o G/K$, where o denotes the identity coset of G/K (cf. [3], [7]). Under this identification, the isotropy representation $\chi : K \rightarrow \text{Aut}(\mathfrak{m})$ of G/K (or simply K) is equivalent to the adjoint representation $\text{Ad}|_K$ restricted on \mathfrak{m} , i.e. $\chi(k) = \text{Ad}(k)|_{\mathfrak{m}}$ for all $k \in K$. Thus the set of all G -invariant symmetric covariant 2-tensors on G/K is identified with the set of all $\text{Ad}(K)$ -invariant symmetric bilinear forms on \mathfrak{m} . In particular, the set of G -invariant Riemannian metrics on G/K is identified with the set of $\text{Ad}(K)$ -invariant inner products $\langle \cdot, \cdot \rangle$ on \mathfrak{m} (cf. [3], [7]).

Let now g be a G -invariant metric on $M = G/K$. Since G is compact and semisimple, the Ricci tensor Ric_g of g is given by ([7, 7.38]):

$$\text{Ric}_g(X, X) = -\frac{1}{2} \sum_i |[X, X_i]_{\mathfrak{m}}|^2 + \frac{1}{2}(X, X) + \frac{1}{4} \sum_{i,j} \langle [X_i, X_j]_{\mathfrak{m}}, X \rangle^2, \quad (3)$$

where $\{X_i\}$ is an orthonormal basis of $\mathfrak{m} = T_o G/K$ with respect to the $\text{Ad}(K)$ -invariant inner product $\langle \cdot, \cdot \rangle$, induced by the G -invariant metric g . Recall that we can define an $\text{Ad}(K)$ -equivariant, g -selfadjoint endomorphism ric_g , through the relation $\text{Ric}_g(\cdot, \cdot) = g(\text{ric}_g(\cdot), \cdot)$. This operator is known as the Ricci operator corresponding to g . Finally, the scalar curvature

$S_g = \text{tr Ric}_g$ of g is given by ([7, 7.39]):

$$S_g = \frac{1}{2} \sum_i (X_i, X_i) - \frac{1}{4} \sum_{i,j} |[X_i, X_j]_{\mathfrak{m}}|^2. \quad (4)$$

Note that the scalar curvature S_g is a constant function on $M = G/K$, and so

$$r = \frac{\int_M S_g du_g}{\int_M du_g} = \frac{S_g \int_M du_g}{\int_M du_g} = S_g.$$

Thus for a G -invariant metric g on $M = G/K$ the normalized Ricci flow (2) is equivalent to $\frac{\partial g}{\partial t} = -2 \text{ric}_g + \frac{2S_g}{n} \cdot g$, where $n = \dim G/K$ (cf. [8]).

We now assume that $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ is a (\cdot, \cdot) -orthogonal decomposition of \mathfrak{m} into irreducible mutually inequivalent $\text{Ad}(K)$ -submodules \mathfrak{m}_j , that is $\mathfrak{m}_i \not\cong \mathfrak{m}_j$ (as $\text{Ad}(K)$ -representations), for any $1 \leq i \neq j \leq s$. Such a decomposition always exists for any flag manifold $M = G/K$ and it is expressed in terms of \mathfrak{t} -roots (cf. [2], [3], [6]). It enables us to parametrize the set \mathcal{M}^G of G -invariant Riemannian metrics g on G/K (or equivalently the set of $\text{Ad}(K)$ -invariant inner products $\langle \cdot, \cdot \rangle$ on \mathfrak{m}), as follows:

$$g = \langle \cdot, \cdot \rangle = x_1 \cdot (\cdot, \cdot)|_{\mathfrak{m}_1} + \cdots + x_s \cdot (\cdot, \cdot)|_{\mathfrak{m}_s}, \quad (5)$$

where $(x_1, \dots, x_s) \in \mathbb{R}_+^s$. These metrics are diagonal with respect to the decomposition $\mathfrak{m} = \oplus_{i=1}^s \mathfrak{m}_i$ and since $\mathfrak{m}_i \not\cong \mathfrak{m}_j$, Schur's Lemma implies that any G -invariant metric on G/K is of this form (cf. [23]). Similarly, the Ricci tensor Ric_g , as a G -invariant symmetric covariant 2-tensor on G/K , is identified with an $\text{Ad}(K)$ -invariant symmetric bilinear form on \mathfrak{m} and it is given by $\text{Ric}_g = \sum_{i=1}^s r_i \cdot (\cdot, \cdot)|_{\mathfrak{m}_i}$, where r_1, \dots, r_s are the components of the Ricci tensor on each \mathfrak{m}_i . Thus the Ricci operator ric_g has the form $\text{ric}_g = \sum_{i=1}^s (x_i \cdot r_i) \cdot (\cdot, \cdot)|_{\mathfrak{m}_i}$, where x_1, \dots, x_s are the components of the metric tensor g .

Let now $\{e_\alpha\}$ be a (\cdot, \cdot) -orthonormal basis adapted to the isotropy decomposition $\mathfrak{m} = \oplus_{i=1}^s \mathfrak{m}_i$, that is $e_\alpha \in \mathfrak{m}_i$ for some i , and $\alpha < \beta$ if $i < j$ (with $e_\alpha \in \mathfrak{m}_i$ and $e_\beta \in \mathfrak{m}_j$). Let $A_{\alpha\beta}^\gamma = ([e_\alpha, e_\beta], e_\gamma)$, so that $[e_\alpha, e_\beta]_{\mathfrak{m}} = \sum_\gamma A_{\alpha\beta}^\gamma e_\gamma$, and set

$$\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A_{\alpha\beta}^\gamma)^2 = \sum ([e_\alpha, e_\beta], e_\gamma)^2, \quad (6)$$

where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i, e_\beta \in \mathfrak{m}_j$, and $e_\gamma \in \mathfrak{m}_k$. These triples are called the *structure constants of G/K* with respect to the decomposition $\mathfrak{m} = \oplus_{i=1}^s \mathfrak{m}_i$. Note that $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the (\cdot, \cdot) -orthonormal bases chosen for $\mathfrak{m}_i, \mathfrak{m}_j$ and \mathfrak{m}_k , but it depends on the choice of the decomposition of \mathfrak{m} ([23]). Also the structure constants are non-negative, that is $\begin{bmatrix} k \\ ij \end{bmatrix} \geq 0$ and $\begin{bmatrix} k \\ ij \end{bmatrix} = 0$, if and only if $([\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_k) = 0$ and they are symmetric in all three entries, that is $\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}$. By reconstructing an $\langle \cdot, \cdot \rangle$ -orthonormal basis for each isotropy summand \mathfrak{m}_k and applying relations (3) and (4), we obtain the following useful expressions.

Proposition 1. ([23], [20]) *Let $M = G/K$ be a generalized flag manifold of a compact semisimple Lie group G and let $\mathfrak{m} = \oplus_{i=1}^s \mathfrak{m}_i$ be an isotropy decomposition of \mathfrak{m} . Let g be a G -invariant metric on M defined by (5), and set $d_k = \dim \mathfrak{m}_k$ for all $k = 1, \dots, s$. Then:*

(1) *The components r_1, \dots, r_s of the Ricci tensor Ric_g are given by*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{i,j} \frac{x_k}{x_i x_j} \begin{bmatrix} k \\ ij \end{bmatrix} - \frac{1}{2d_k} \sum_{i,j} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ ki \end{bmatrix}, \quad (k = 1, \dots, s).$$

(2) The scalar curvature $S_g = \text{tr Ric}_g = \sum_{i=1}^s d_i r_i$ is given by $S = \frac{1}{2} \sum_{i=1}^s \frac{d_i}{x_i} - \frac{1}{4} \sum_{i,j,k} \begin{bmatrix} k \\ ij \end{bmatrix} \frac{x_k}{x_i x_j}$.

It follows that for a G -invariant metric $g = \langle \cdot, \cdot \rangle$ given by (5), the normalized Ricci flow reduces to the following system (we omit the minus sign of the Ricci operator since our G -invariant metric $g = \langle \cdot, \cdot \rangle$ is expressed with respect to the inner product $(\cdot, \cdot) = -B(\cdot, \cdot)$):

$$\left. \begin{aligned} \dot{x}_1 &= 2x_1 \cdot r_1 + \frac{2S_g}{n} \cdot x_1, \\ \dot{x}_2 &= 2x_2 \cdot r_2 + \frac{2S_g}{n} \cdot x_2, \\ &\vdots \\ \dot{x}_s &= 2x_s \cdot r_s + \frac{2S_g}{n} \cdot x_s. \end{aligned} \right\} \quad (7)$$

2. FLAG MANIFOLDS WITH TWO AND THREE ISOTROPY SUMMANDS

For the construction of flag manifolds with two or three isotropy summands, it is useful to describe the structure of such a space $M = G/K = G/C(S)$ in terms of Lie theory. By using this description one can classify flag manifolds through the *painted Dynkin diagrams*.

2.1. A description of flag manifolds in terms of painted Dynkin diagrams. For simplicity we assume that G is simple with Lie algebra \mathfrak{g} . Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_{\alpha}^{\mathbb{C}}$ be the root space decomposition of the complexification $\mathfrak{g}^{\mathbb{C}}$ of \mathfrak{g} , with respect to a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$, where $R \subset (\mathfrak{h}^{\mathbb{C}})^*$ is the root system of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{g}_{\alpha}^{\mathbb{C}} = \{X \in \mathfrak{g}^{\mathbb{C}} : \text{ad}(H)X = \alpha(H)X, \text{ for all } H \in \mathfrak{h}^{\mathbb{C}}\}$ are the 1-dimensional root spaces. As usual, we identify $(\mathfrak{h}^{\mathbb{C}})^*$ with $\mathfrak{h}^{\mathbb{C}}$ via the Killing form B of $\mathfrak{g}^{\mathbb{C}}$. Let $\Pi = \{\alpha_1, \dots, \alpha_{\ell}\}$ ($\dim \mathfrak{h}^{\mathbb{C}} = \ell$) be a fundamental system of R and choose a subset Π_K of Π . We denote by $R_K = \{\beta \in R : \beta = \sum_{\alpha_i \in \Pi_K} k_i \alpha_i\}$ the closed subsystem spanned by Π_K . Then the Lie subalgebra $\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\beta \in R_K} \mathfrak{g}_{\beta}^{\mathbb{C}}$ is a reductive subalgebra of $\mathfrak{g}^{\mathbb{C}}$, i.e. it admits a decomposition of the form $\mathfrak{k}^{\mathbb{C}} = Z(\mathfrak{k}^{\mathbb{C}}) \oplus \mathfrak{k}_{ss}^{\mathbb{C}}$, where $Z(\mathfrak{k}^{\mathbb{C}})$ is its center and $\mathfrak{k}_{ss}^{\mathbb{C}} = [\mathfrak{k}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}]$ the semisimple part of $\mathfrak{k}^{\mathbb{C}}$. In particular, R_K is the root system of $\mathfrak{k}_{ss}^{\mathbb{C}}$, and thus Π_K can be considered as the associated fundamental system. Let K be the connected Lie subgroup of G generated by $\mathfrak{k} = \mathfrak{k}^{\mathbb{C}} \cap \mathfrak{g}$. Then the homogeneous manifold $M = G/K$ is a flag manifold, and any flag manifold is defined in this way, i.e., by the choice of a triple $(\mathfrak{g}^{\mathbb{C}}, \Pi, \Pi_K)$ (cf. [1], [2]).

Set $\Pi_M = \Pi \setminus \Pi_K$ and $R_M = R \setminus R_K$, such that $\Pi = \Pi_K \sqcup \Pi_M$, and $R = R_K \sqcup R_M$, respectively. Roots in R_M are called *complementary roots*, and they possess an important role in the geometry of $M = G/K$. For example, if $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$ is a reductive decomposition of $\mathfrak{g}^{\mathbb{C}}$, then $\mathfrak{m}^{\mathbb{C}} = (T_o G/K)^{\mathbb{C}} = \sum_{\alpha \in R_M} \mathfrak{g}_{\alpha}^{\mathbb{C}}$.

Definition 2. Let $\Gamma = \Gamma(\Pi)$ be the Dynkin diagram of the fundamental system Π . By painting in black the nodes of Γ corresponding to Π_M , we obtain the painted Dynkin diagram of the flag manifold G/K . In this diagram the subsystem Π_K is determined as the subdiagram of white roots.

Conversely, given a painted Dynkin diagram, in order to determine the associated flag manifold $M = G/K$ we are working as follows: At first, we obtain the group G , as the unique simply connected Lie group generated by the unique real form \mathfrak{g} of the complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ (up to inner automorphisms of $\mathfrak{g}^{\mathbb{C}}$ [17, p. 184]), which is reconstructed by the underlying Dynkin diagram. On the other hand, the connected Lie subgroup $K \subset G$ is defined by using the additional information $\Pi = \Pi_K \cup \Pi_M$ coded into the painted Dynkin diagram. The semisimple

part of K , is obtained from the (not necessarily connected) subdiagram of white roots, and each black root, i.e. each root in Π_M , gives rise to one $U(1)$ -summand. Thus the painted Dynkin diagram determines the isotropy group K and the space $M = G/K$ completely. By using certain rules to determine whether different painted Dynkin diagrams define isomorphic flag manifolds (see [1], [2]), one can obtain all flag manifolds G/K of a compact simple Lie group G .

2.2. Flag manifolds with two isotropy summands. Let G be a compact simple Lie group. We will construct the normalized Ricci flow equation for G -invariant metrics on flag manifolds $M = G/K$ with two isotropy summands, that is $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. These spaces have been classified in terms of painted Dynkin diagrams in [4] (see also [5], [22]), and they are given in Table 1. In particular, they are determined by painting black in the Dynkin diagram of G , a simple root $\alpha_p \in \Pi$ with height 2, that is $\Pi \setminus \Pi_K = \Pi_M = \{\alpha_p : \text{ht}(\alpha_p) = 2\}$.¹

Table 1. The generalized flag manifolds with two isotropy summands.

G simple	Flag manifold G/K with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$
B_ℓ	$SO(2\ell + 1)/U(p) \times SO(2(\ell - p) + 1)$ ($2 \leq p \leq \ell$)
C_ℓ	$Sp(\ell)/U(p) \times Sp(\ell - p)$ ($1 \leq p \leq \ell - 1$)
D_ℓ	$SO(2\ell)/U(p) \times SO(2(\ell - p))$ ($2 \leq p \leq \ell - 2$)
G_2	$G_2/U(2)$ ($U(2)$ is represented by the short root of G_2)
F_4	$F_4/SO(7) \times U(1)$ $F_4/Sp(3) \times U(1)$
E_6	$E_6/SU(6) \times U(1)$ $E_6/SU(2) \times SU(5) \times U(1)$
E_7	$E_7/SU(7) \times U(1)$ $E_7/SU(2) \times SO(10) \times U(1)$ $E_7/SO(12) \times U(1)$
E_8	$E_8/E_7 \times U(1)$ $E_8/SO(14) \times U(1)$

We mention here, for future use, that the isotropy summands \mathfrak{m}_1 and \mathfrak{m}_2 satisfy the following useful inclusions (see [5], or [11]):

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{k} \oplus \mathfrak{m}_2, \quad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1, \quad [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{k}. \quad (8)$$

Now, according to (5), a G -invariant metric g of M is determined by two positive variables, i.e.

$$g = \langle \cdot, \cdot \rangle = x_1 \cdot (\cdot, \cdot)|_{\mathfrak{m}_1} + x_2 \cdot (\cdot, \cdot)|_{\mathfrak{m}_2}, \quad x_1 > 0, \quad x_2 > 0. \quad (9)$$

Thus the space \mathcal{M}^G of G -invariant metrics on G/K is 2-dimensional (since $\mathfrak{m}_1 \not\cong \mathfrak{m}_2$). From inclusions (8) and by applying relation (6), we easily conclude that the only non-zero structure constants $\begin{bmatrix} k \\ ji \end{bmatrix}$ of G/K are the triples $\begin{bmatrix} 2 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ 21 \end{bmatrix} \neq 0$. Now the Ricci tensor Ric_g of (M, g) is given by $\text{Ric}_g = r_1 \cdot (\cdot, \cdot)|_{\mathfrak{m}_1} + r_2 \cdot (\cdot, \cdot)|_{\mathfrak{m}_2}$, where $r_1, r_2 \in \mathbb{R}$. Thus by applying Proposition 1 we easily obtain

Proposition 2. *Let $M = G/K$ be a generalized flag manifold with two isotropy summands and let g be a G -invariant Riemannian metric on M given by (9).*

(1) *The components r_1, r_2 of the Ricci tensor Ric_g are given as follows:*

$$\begin{cases} r_1 = \frac{1}{2x_1} - \begin{bmatrix} 2 \\ 11 \end{bmatrix} \frac{x_2}{2d_1 x_1^2}, \\ r_2 = \frac{1}{2x_2} + \begin{bmatrix} 2 \\ 11 \end{bmatrix} \left(\frac{x_2}{4d_2 x_1^2} - \frac{1}{2d_2 x_2} \right). \end{cases}$$

¹The height of a simple root $\alpha_p \in \Pi$ ($p = 1, \dots, \ell$), is the positive integer c_p in the expression of the highest root $\tilde{\alpha} = \sum_{k=1}^{\ell} c_k \alpha_k$ of $\mathfrak{g}^{\mathbb{C}}$, in terms of simple roots. Note that $\text{ht} : \Pi \rightarrow \mathbb{Z}^+$ is the function defined by $\text{ht}(\alpha_p) = c_p$.

(2) The scalar curvature S_g is given by $S_g = \sum_{i=1}^2 d_i \cdot r_i = \frac{1}{2} \left(\frac{d_1}{x_1} + \frac{d_2}{x_2} \right) - \frac{1}{4} \begin{bmatrix} 2 \\ 11 \end{bmatrix} \left(\frac{x_2}{x_1^2} + 2 \frac{1}{x_2} \right)$.

In order to find the non-zero structure constant $\begin{bmatrix} 2 \\ 11 \end{bmatrix}$ of $M = G/K$ (with respect the decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$), we use the unique G -invariant Kähler-Einstein metric g_J which admits $M = G/K$, compatible with the unique G -invariant complex structure J on M (cf. [9, 13.8]). This invariant metric is given by (cf. [5])

$$g_J = 1 \cdot (\cdot, \cdot)|_{\mathfrak{m}_1} + 2 \cdot (\cdot, \cdot)|_{\mathfrak{m}_2}.$$

Substituting the values $x_1 = 1$, and $x_2 = 2$ in $r_1 - r_2 = 0$, we easily obtain that (cf. [5])

$$\begin{bmatrix} 2 \\ 11 \end{bmatrix} = \frac{d_1 d_2}{d_1 + 4d_2}.$$

The dimensions $d_i = \dim \mathfrak{m}_i$ were calculated in [5] and are given in the following table:

Table 2. The dimensions $d_i = \dim \mathfrak{m}_i$ for any $M = G/K$ with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$

$M = G/K$	d_1	d_2
$SO(2\ell+1)/U(p) \times SO(2(\ell-p)+1) \quad (2 \leq p \leq \ell)$	$2p(2(\ell-p)+1)$	$p(p-1)$
$Sp(\ell)/U(p) \times Sp(\ell-p) \quad (1 \leq p \leq \ell-1)$	$4p(\ell-p)$	$p(p+1)$
$SO(2\ell)/U(p) \times SO(2(\ell-p)) \quad (2 \leq p \leq \ell-2)$	$4p(\ell-p)$	$p(p-1)$
$G_2/U(2)$	8	2
$F_4/SO(7) \times U(1)$	16	14
$F_4/Sp(3) \times U(1)$	28	2
$E_6/SU(6) \times U(1)$	40	2
$E_6/SU(2) \times SU(5) \times U(1)$	40	10
$E_7/SU(7) \times U(1)$	70	14
$E_7/SU(2) \times SO(10) \times U(1)$	64	20
$E_7/SO(12) \times U(1)$	64	2
$E_8/E_7 \times U(1)$	112	2
$E_8/SO(14) \times U(1)$	128	28

Now, the Ricci components r_1, r_2 and the scalar curvature S_g are completely determined from Proposition 2. As a consequence of relation (7) we obtain that the normalized Ricci flow for a homogeneous initial metric (9) on $M = G/K$ is given by

$$\left. \begin{aligned} \dot{x}_1 &= 2x_1 \cdot r_1 + \frac{2x_1}{d_1 + d_2} \cdot S_g, \\ \dot{x}_2 &= 2x_2 \cdot r_2 + \frac{2x_2}{d_1 + d_2} \cdot S_g. \end{aligned} \right\} \quad (10)$$

Remark 1. In [13] Dickinson and Kerr applied the variational method to show that the number of G -invariant Einstein metrics on $M = G/K$ is two. The explicit form of these metrics (which are not isometric) was given in [5] by solving the equation $r_1 - r_2 = 0$, substituting first the above value of the triple $\begin{bmatrix} 2 \\ 11 \end{bmatrix}$. The first one is the above defined Kähler-Einstein metric, and the second one is non-Kähler and it is given by

$$g = 1 \cdot (\cdot, \cdot)|_{\mathfrak{m}_1} + \frac{4d_2}{d_1 + 2d_2} \cdot (\cdot, \cdot)|_{\mathfrak{m}_2}.$$

2.3. Flag manifolds with three isotropy summands. Flag manifolds $M = G/K$ of a compact simple Lie group G , whose isotropy representation decomposes into three pairwise non isomorphic irreducible $\text{Ad}(K)$ -modules, i.e. $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, are obtained by painting the Dynkin diagram $\Gamma(\Pi)$ of G with two different ways. We can either paint a simple root with height 3, or two simple roots both of height 1. Thus the pairs (Π, Π_K) which determine such spaces are divided into two different types as follows (cf. [12]):

Type	(Π, Π_K)
I	$\Pi \setminus \Pi_K = \{\alpha_p : \text{ht}(\alpha_p) = 3\}$
II	$\Pi \setminus \Pi_K = \{\alpha_p, \alpha_q \ (1 \leq p \neq q \leq \ell) : \text{ht}(\alpha_p) = \text{ht}(\alpha_q) = 1\}$

In the following we shall call the flag manifolds determined by a pair (Π, Π_K) of Type I (resp. of Type II), *flag manifolds of Type I* (resp. *flag manifolds of Type II*). We present these homogeneous spaces in Table 3.

Table 3. The generalized flag manifolds with three isotropy summands.

G simple	Flag manifold	G/K of Type I
E_8	$E_8/E_6 \times SU(2) \times U(1)$	
	$E_8/SU(8) \times U(1)$	
E_7	$E_7/SU(5) \times SU(3) \times U(1)$	
	$E_7/SU(6) \times SU(2) \times U(1)$	
E_6	$E_6/SU(3) \times SU(3) \times SU(2) \times U(1)$	
F_4	$F_4/SU(3) \times SU(2) \times U(1)$	$(SU(2)$ is represented by the long root of F_4)
G_2	$G_2/U(2)$	$(U(2)$ is represented by the long root of G_2)
G simple	Flag manifold	G/K of Type II
A_ℓ	$SU(\ell + m + n)/S(U(\ell) \times U(m) \times U(n))$	$(\ell, m, n \in \mathbb{Z}^+)$
D_ℓ	$SO(2\ell)/U(1) \times U(\ell - 1)$	$(\ell \geq 4)$
E_6	$E_6/SO(8) \times U(1) \times U(1)$	

2.4. Flag manifolds of Type I. Let us now construct the normalized Ricci flow for a flag manifold $M = G/K$ of Type I with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$. The case of the flag manifolds of Type II will be treated in a forthcoming paper (see also [16] for the full flag manifold $SU(3)/T$).

The dimensions $d_i = \dim \mathfrak{m}_i$ ($i = 1, 2, 3$) have been calculated in [12, p. 311], and are presented in Table 4.

Table 4. The dimensions $d_i = \dim \mathfrak{m}_i$ for any $M = G/K$ of Type I.

Flag manifold	G/K of Type I	d_1	d_2	d_3
$E_8/E_6 \times SU(2) \times U(1)$		108	54	4
$E_8/SU(8) \times U(1)$		112	56	16
$E_7/SU(5) \times SU(3) \times U(1)$		60	30	8
$E_7/SU(6) \times SU(2) \times U(1)$		60	30	4
$E_6/SU(3) \times SU(3) \times SU(2) \times U(1)$		36	18	4
$F_4/SU(3) \times SU(2) \times U(1)$		24	12	4
$G_2/U(2)$		4	2	4

Note that the isotropy summands $\mathfrak{m}_1, \mathfrak{m}_2$ and \mathfrak{m}_3 fullfil the following inclusions (see [11]):

$$\left. \begin{aligned} [\mathfrak{m}_1, \mathfrak{m}_1] &\subset \mathfrak{k} \oplus \mathfrak{m}_2, & [\mathfrak{m}_1, \mathfrak{m}_2] &\subset \mathfrak{m}_1 \oplus \mathfrak{m}_3, & [\mathfrak{m}_1, \mathfrak{m}_3] &\subset \mathfrak{m}_2, \\ [\mathfrak{m}_2, \mathfrak{m}_2] &\subset \mathfrak{k}, & [\mathfrak{m}_2, \mathfrak{m}_3] &\subset \mathfrak{m}_1, & [\mathfrak{m}_3, \mathfrak{m}_3] &\subset \mathfrak{k}. \end{aligned} \right\} \quad (11)$$

According to (5), a G -invariant metric g of M is determined by three positive parameters, i.e.

$$g = \langle \cdot, \cdot \rangle = x_1 \cdot (\cdot, \cdot)|_{\mathfrak{m}_1} + x_2 \cdot (\cdot, \cdot)|_{\mathfrak{m}_2} + x_3 \cdot (\cdot, \cdot)|_{\mathfrak{m}_3}, \quad x_1 > 0, \ x_2 > 0, \ x_3 > 0, \quad (12)$$

and since $\mathfrak{m}_1 \not\cong \mathfrak{m}_2 \not\cong \mathfrak{m}_3$, the space \mathcal{M}^G of G -invariant metrics on G/K is 3-dimensional. In view of (11) and by applying relation (6), we conclude that the only non zero structure constants of $M = G/K$ are the following

$$\begin{aligned} \begin{bmatrix} 1 \\ 12 \end{bmatrix} &= \begin{bmatrix} 1 \\ 21 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix} = c_{11}^2 \\ \begin{bmatrix} 3 \\ 12 \end{bmatrix} &= \begin{bmatrix} 3 \\ 21 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \end{bmatrix} = \begin{bmatrix} 2 \\ 31 \end{bmatrix} = \begin{bmatrix} 1 \\ 23 \end{bmatrix} = \begin{bmatrix} 1 \\ 32 \end{bmatrix} = c_{12}^3. \end{aligned} \quad (13)$$

Now, the Ricci tensor Ric_g of g is given by $\text{Ric}_g = r_1 \cdot (\cdot, \cdot)|_{\mathfrak{m}_1} + r_2 \cdot (\cdot, \cdot)|_{\mathfrak{m}_2} + r_3 \cdot (\cdot, \cdot)|_{\mathfrak{m}_3}$, where $r_1, r_2, r_3 \in \mathbb{R}$. Thus, by applying Proposition 1, we easily obtain the following.

Proposition 3. *Let $M = G/K$ be a generalized flag manifold with three isotropy summands of Type I and let g be a G -invariant Riemannian metric on M given by (12).*

(1) *The components r_1, r_2, r_3 of the Ricci tensor Ric_g are given as follows:*

$$\begin{cases} r_1 = \frac{1}{2x_1} - \frac{c_{11}^2 x_2}{2d_1 x_1^2} + \frac{c_{12}^3}{2d_1} \left(\frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} - \frac{x_3}{x_1 x_2} \right), \\ r_2 = \frac{1}{2x_2} + \frac{c_{11}^2}{4d_2} \left(\frac{x_2}{x_1^2} - \frac{2}{x_2} \right) + \frac{c_{12}^3}{2d_2} \left(\frac{x_2}{x_1 x_3} - \frac{x_1}{x_2 x_3} - \frac{x_3}{x_1 x_2} \right), \\ r_3 = \frac{1}{2x_3} + \frac{c_{12}^3}{2d_3} \left(\frac{x_3}{x_1 x_2} - \frac{x_1}{x_2 x_3} - \frac{x_2}{x_1 x_3} \right). \end{cases}$$

(2) *The scalar curvature S_g is given by*

$$S_g = \sum_{i=1}^3 d_i \cdot r_i = \frac{1}{2} \left(\frac{d_1}{x_1} + \frac{d_2}{x_2} + \frac{d_3}{x_3} \right) - \frac{c_{11}^2}{4} \left(\frac{x_2}{x_1^2} + \frac{2}{x_2} \right) - \frac{c_{12}^3}{2} \left(\frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} + \frac{x_3}{x_1 x_2} \right).$$

For the computation of the unknown triples c_{11}^2 and c_{12}^3 we use the unique G -invariant Kähler-Einstein metric which admits any flag manifold G/K of Type I, and given by (cf. [12]):

$$g_J = \langle \cdot, \cdot \rangle = 1 \cdot (\cdot, \cdot)|_{\mathfrak{m}_1} + 2 \cdot (\cdot, \cdot)|_{\mathfrak{m}_2} + 3 \cdot (\cdot, \cdot)|_{\mathfrak{m}_3}.$$

Substituting the values $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$ in the system

$$r_1 - r_2 = 0, \quad r_2 - r_3 = 0, \tag{14}$$

where r_1, r_2, r_3 are given by Proposition 3, we obtain the following values:

$$c_{11}^2 = \frac{d_1 d_2 + 2d_1 d_3 - d_2 d_3}{d_1 + 4d_2 + 9d_3}, \quad c_{12}^3 = \frac{(d_1 + d_2)d_3}{d_1 + 4d_2 + 9d_3}. \tag{15}$$

Thus, the Ricci components r_1, r_2, r_3 and the scalar curvature S_g are completely determined from Proposition 3. By applying relation (7), we obtain the normalized Ricci flow for a homogeneous initial metric (12) on $M = G/K$, defined by the following system:

$$\left. \begin{aligned} \dot{x}_1 &= 2x_1 \cdot r_1 + \frac{2x_1}{d_1 + d_2 + d_3} \cdot S_g, \\ \dot{x}_2 &= 2x_2 \cdot r_2 + \frac{2x_2}{d_1 + d_2 + d_3} \cdot S_g, \\ \dot{x}_3 &= 2x_3 \cdot r_3 + \frac{2x_3}{d_1 + d_2 + d_3} \cdot S_g. \end{aligned} \right\} \tag{16}$$

Remark 2. In [12], was proved that the number of G -invariant Einstein metrics on a flag manifold $M = G/K$ of Type I is exactly three. One is the above defined Kähler-Einstein metric g_J and the other two are non Kähler. The explicit form however of these two metrics was not presented. We give here these explicit forms by solving equation (14). First we normalize the G -invariant metric (12) by setting $x_1 = 1$. We obtain the following theorem:

Theorem 1. ([12]) *Let $M = G/K$ be a generalized flag manifold of a compact simple Lie group G of Type I. Then M admits precisely three (up to a scale) G -invariant Einstein metrics. One is Kähler-Einstein given by $g_J = (1, 2, 3)$, and the other two are non Kähler metrics g_1, g_2 , given below:*

Flag manifold G/K of Type I	$g_1 = (1, x_2, x_3)$	$g_2 = (1, x_2, x_3)$
$E_8/E_6 \times SU(2) \times U(1)$	(1, 0.914286, 1.54198)	(1, 1.0049, 0.129681)
$E_8/SU(8) \times U(1)$	(1, 0.717586, 1.25432)	(1, 1.06853, 0.473177)
$E_7/SU(5) \times SU(3) \times U(1)$	(1, 0.733552, 1.27681)	(1, 1.06029, 0.443559)
$E_7/SU(6) \times SU(2) \times U(1)$	(1, 0.85368, 1.45259)	(1, 1.01573, 0.229231)
$E_6/SU(3) \times SU(3) \times SU(2) \times U(1)$	(1, 0.771752, 1.33186)	(1, 1.04268, 0.373467)
$F_4/SU(3) \times SU(2) \times U(1)$	(1, 0.678535, 1.20122)	(1, 1.09057, 0.546045)
$G_2/U(2)$	(1, 1.67467, 2.05238)	(1, 0.186894, 0.981478)

3. THE METHOD OF POINCARÉ COMPACTIFICATION

Our aim is to calculate the Einstein metrics presented above once again, using the fact that they correspond to fixed points of the normalized Ricci flow. As we shall see, the fixed points of this flow are located at infinity and not in the finite space. The study of a vector field at infinity is possible by making use of the compactification procedure due to Poincaré ([21]). To make more clear the arguments that follow, we briefly recall the two-dimensional case here, providing also the formulas for the three-dimensional case we will need in the subsequent calculations. For details on the general case we refer the reader to [15].

To study a (polynomial) vector field in a neighborhood of infinity, we introduce a new vector field, defined on a sphere, as follows. Let (x_1, x_2) be coordinates on \mathbb{R}^2 and $X = P(x_1, x_2)\frac{\partial}{\partial x_1} + Q(x_1, x_2)\frac{\partial}{\partial x_2}$ a polynomial vector field of degree d (that is, $d = \max\{\deg(P), \deg(Q)\}$). If (y_1, y_2, y_3) denote the coordinates on \mathbb{R}^3 then we consider \mathbb{R}^2 to be the plane of \mathbb{R}^3 defined as $(y_1, y_2, y_3) = (x_1, x_2, 1)$. We consider also the sphere $\mathbb{S}^2 = \{y \in \mathbb{R}^3 / y_1^2 + y_2^2 + y_3^2 = 1\}$, which we shall call Poincaré sphere. This sphere is divided to the northern ($H^+ = \{y \in \mathbb{S}^2 / y_3 > 0\}$) and to the southern ($H^- = \{y \in \mathbb{S}^2 / y_3 < 0\}$) hemispheres, and the equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 / y_3 = 0\}$.

The central projections from \mathbb{R}^2 to the Poincaré sphere are defined as follows:

$$f^+ : \mathbb{R}^2 \rightarrow \mathbb{S}^2, (x_1, x_2) \mapsto \left(\frac{x_1}{\Delta(x)}, \frac{x_2}{\Delta(x)}, \frac{1}{\Delta(x)}\right) \text{ and } \\ f^- : \mathbb{R}^2 \rightarrow \mathbb{S}^2, (x_1, x_2) \mapsto \left(\frac{-x_1}{\Delta(x)}, \frac{-x_2}{\Delta(x)}, \frac{-1}{\Delta(x)}\right),$$

where $\Delta(x) = \sqrt{x_1^2 + x_2^2 + 1}$, and in this way we obtain one vector field on each hemisphere. Each one of these vector fields, namely

$$\bar{X}(y) = D_x f^+(X(x)), \quad y = f^+(x) \quad \text{and} \quad \bar{X}(y) = D_x f^-(X(x)), \quad y = f^-(x),$$

is conjugate to the original vector field. We have thus constructed a vector field \bar{X} on $\mathbb{S}^2 \setminus \mathbb{S}^1$ and we want to extend it to \mathbb{S}^2 . To achieve this, we multiply the vector field by the function $\rho(y) = y_3^{d-1}$. We state, without proof, the following theorem ([15]):

Theorem 2. *The field \bar{X} can be analytically extended to the whole sphere by multiplication with the factor y_3^{d-1} , in such a way that the equator is invariant.*

The vector field defined on the Poincaré sphere is called the Poincaré compactification of the original vector field $X = (P, Q)$ and is denoted by $p(X)$. Points of the equator correspond to the points at infinity of the plane.

To study the vector field $p(X)$ we make use of six local charts on the Poincaré sphere, given by $U_k = \{y \in \mathbb{S}^2 / y_k > 0\}$ and $V_k = \{y \in \mathbb{S}^2 / y_k < 0\}$, $k = 1, 2, 3$. The local maps for the corresponding charts are given by

$$\phi_k : U_k \rightarrow \mathbb{R}^2 \text{ and } \psi_k : V_k \rightarrow \mathbb{R}^2,$$

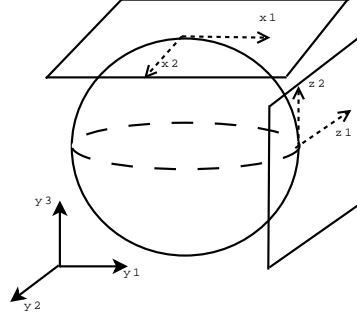


FIGURE 1. The Poincaré sphere, \mathbb{R}^2 as its tangent space at the north pole and the U_1 chart (vertical plane).

with $\phi_k(y) = -\psi_k(y) = (y_m/y_k, y_n/y_k)$, for $m < n$ and $m, n \neq k$. If we write $z = (z_1, z_2)$ for the value of $\phi_k(y)$ or $\psi_k(y)$ then, in any chart, points at infinity correspond to $z_2 = 0$. Note that the meaning of z depends on the chart.

We will now write down the expressions of $p(X)$ in the local charts, for future reference. In the chart (U_1, ϕ_1) (corresponding in Figure 1 as the $y_1 = 1$ plane) the expression of the field reads as:

$$\begin{aligned} \dot{z}_1 &= z_2^d [-z_1 P(\frac{1}{z_2}, \frac{z_1}{z_2}) + Q(\frac{1}{z_2}, \frac{z_1}{z_2})] \\ \dot{z}_2 &= -z_2^{d+1} P(\frac{1}{z_2}, \frac{z_1}{z_2}). \end{aligned} \quad (17)$$

The expression in the chart (U_2, ϕ_2) is

$$\begin{aligned} \dot{z}_1 &= z_2^d [P(\frac{z_1}{z_2}, \frac{1}{z_2}) - z_1 Q(\frac{z_1}{z_2}, \frac{1}{z_2})] \\ \dot{z}_2 &= -z_2^{d+1} Q(\frac{z_1}{z_2}, \frac{1}{z_2}), \end{aligned}$$

while for (U_3, ϕ_3) is

$$\begin{aligned} \dot{z}_1 &= P(z_1, z_2) \\ \dot{z}_2 &= Q(z_1, z_2). \end{aligned}$$

We omit the expressions of $p(X)$ in the charts (V_k, ψ_k) , since they coincide with the expressions for (U_k, ϕ_k) multiplied by the factor $(-1)^{d-1}$ for $k = 1, 2, 3$.

If one is interested to study the global behavior of a vector field on \mathbb{R}^2 , that is, if we are also interested to study the vector field in a neighborhood of infinity, then clearly, it is enough to work on $H^+ \cup \mathbb{S}^1$, which is called the Poincaré disk.

The above described procedure generalizes to every dimension. Since we also need the three-dimensional case here we include now the necessary formulas of the compactified vector field, using again (z_1, z_2, z_3) as coordinates. If the original vector field is $X = (P^1, P^2, P^3)$, then the equations of the compactified field $p(X)$ read as:

$$\begin{aligned} \dot{z}_1 &= Q^1(z_1, z_2, z_3) = Q^1 \\ \dot{z}_2 &= Q^2(z_1, z_2, z_3) = Q^2 \\ \dot{z}_3 &= Q^3(z_1, z_2, z_3) = Q^3 \end{aligned}$$

where

$$\begin{aligned}
(Q^1, Q^2, Q^3) &= \frac{z_3^d}{(\Delta z)^{d-1}} \left(-z_1 P^1(1/z_3, z_1/z_3, z_2/z_3) + P^2(1/z_3, z_1/z_3, z_2/z_3), \right. \\
&\quad \left. -z_2 P^1(1/z_3, z_1/z_3, z_2/z_3) + P^3(1/z_3, z_1/z_3, z_2/z_3), -z_3 P^1(1/z_3, z_1/z_3, z_2/z_3) \right), \\
(Q^1, Q^2, Q^3) &= \frac{z_3^d}{(\Delta z)^{d-1}} \left(-z_1 P^2(z_1/z_3, 1/z_3, z_2/z_3) + P^1(z_1/z_3, 1/z_3, z_2/z_3), \right. \\
&\quad \left. -z_2 P^2(z_1/z_3, 1/z_3, z_2/z_3) + P^3(z_1/z_3, 1/z_3, z_2/z_3), -z_3 P^2(z_1/z_3, 1/z_3, z_2/z_3) \right), \\
(Q^1, Q^2, Q^3) &= \frac{z_3^d}{(\Delta z)^{d-1}} \left(-z_1 P^3(z_1/z_3, z_2/z_3, 1/z_3) + P^1(z_1/z_3, z_2/z_3, 1/z_3), \right. \\
&\quad \left. -z_2 P^3(z_1/z_3, z_2/z_3, 1/z_3) + P^2(z_1/z_3, z_2/z_3, 1/z_3), -z_3 P^3(z_1/z_3, z_2/z_3, 1/z_3) \right),
\end{aligned}$$

in U_1 , U_2 and U_3 , respectively. The expression for $p(X)$ in U_4 is

$$z_3^{d+1}(P^1(z_1, z_2, z_3), P^2(z_1, z_2, z_3), P^3(z_1, z_2, z_3)),$$

while the expressions at the local charts V_i can be obtained by those at the charts U_i multiplied by the term $(-1)^{d-1}$. By rescaling the time variable, we usually omit the factor $1/(\Delta z)^{d-1}$.

This procedure will be now used, with the aim of analyzing the global behavior of the Ricci flow equations presented in the previous section.

4. THE GLOBAL BEHAVIOUR OF THE NORMALIZED RICCI FLOW

4.1. Dynamics of the Ricci flow on flag manifolds with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. In this section we study the global behavior of the normalized Ricci flow equation for a flag manifold $M = G/K$ with two isotropy summands, namely system (10). This system reduces to:

$$\begin{aligned}
\dot{x}_1 &= \frac{8d_2^2 x_1^2 + 2(2d_1 + d_2)(d_1 + 4d_2)x_1 x_2 - d_2(3d_1 + 2d_2)x_2^2}{2(d_1 + d_2)(d_1 + 4d_2)x_1 x_2} \\
\dot{x}_2 &= \frac{(4d_2 x_1 + d_1 x_2)(4d_2 x_1 + d_1(2x_1 + x_2))}{2(d_1 + d_2)(d_1 + 4d_2)x_1^2}.
\end{aligned}$$

To apply the Poincaré compactification method we multiply this system with the factor $2(d_1 + d_2)(d_1 + 4d_2)x_1^2 x_2$. We remark here that this multiplication will only change the time parametrization of the orbits and not the structure of the phase portrait we wish to determine. We arrive thus to the following two-dimensional system:

$$\left. \begin{aligned} \dot{x}_1 &= 8d_2^2 x_1^2 + 2(2d_1 + d_2)(d_1 + 4d_2)x_1 x_2 - d_2(3d_1 + 2d_2)x_2^2, \\ \dot{x}_2 &= (4d_2 x_1 + d_1 x_2)(4d_2 x_1 + d_1(2x_1 + x_2)), \end{aligned} \right\} \quad (18)$$

the behavior of which, at the first quadrant of the plane, is to be determined. We consider $d_1, d_2 > 0$ as free parameters of the system.

We begin our study with the following lemma.

Lemma 1. *System (18) possesses a single fixed point at the origin. Except from the coordinate axes, the straight lines $\gamma_1(t) = (\frac{1}{2}t, t)$, $\gamma_2(t) = (\frac{d_1+2d_2}{4d_2}t, t)$, remain invariant under its flow.*

We omit the proof of Lemma 1, since it is quite straightforward.

From the dynamical viewpoint, the invariant axes and the lines γ_1, γ_2 consist the separatrices of the parabolic sectors of the fixed point located at the origin (at least the separatrices in the first quadrant). From the Ricci flow viewpoint, the invariance of the axes reflect the fact that no semi-Riemannian or degenerate metric can evolve to a Riemannian metric via the Ricci flow, while, as we shall see, the other two straight lines are related to the Einstein metrics that we are about to compute.

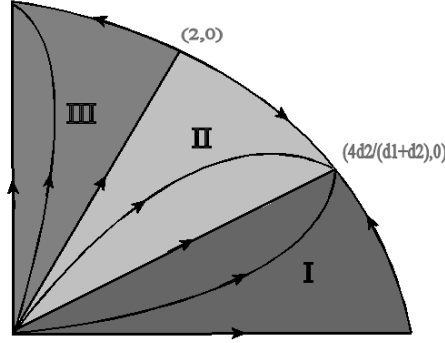


FIGURE 2. System (18) in the first quadrant of the plane.

We proceed now to the study of system (18) at infinity. Since we are only interested in positive values of x_1 and x_2 we restrict ourselves to the (U_1, ϕ_1) chart. Applying (17) and using the notation of the previous section we easily confirm that equations (18), in this chart, take the following form:

$$\begin{aligned}\dot{z}_1 &= (d_1 + d_2)(z_1 - 2)z_1(2d_2(z_1 - 2) + z_1d_1), \\ \dot{z}_2 &= (-4d_1^2z_1 + 3d_1d_2(z_1 - 6)z_1 + 2d_2^2(-4 + (z_1 - 4)z_1))z_2.\end{aligned}$$

By setting $z_2 = 0$ we calculate that there are only two fixed points (except from the origin), namely $(2, 0)$ and $(\frac{4d_2}{d_1+2d_2}, 0)$. The first one is a repelling node and the other one an attracting node. The two invariant lines previously calculated converge to these fixed points, and we are thus able to draw the global phase portrait of the system in Figure 2.

Proposition 4. *The global phase portrait of system (18) is topologically equivalent to the one depicted in Figure (2), when we restrict our attention to the first quadrant of the plane.*

We can now draw conclusions about the existence of Einstein metrics. Recalling that the chart (U_1, ϕ_1) corresponds to the $y_1 = 1$ plane (see the section on the Poincaré compactification), we consider the metrics whose coefficients are determined by the fixed points calculated above:

$$g_1 = 1 \cdot (,)|_{\mathfrak{m}_1} + 2 \cdot (,)|_{\mathfrak{m}_2}, \quad \text{and} \quad g_2 = 1 \cdot (,)|_{\mathfrak{m}_1} + \frac{4d_2}{d_1 + 2d_2} \cdot (,)|_{\mathfrak{m}_2}.$$

We obtain the following:

Theorem 3. *Let $M = G/K$ be a generalized flag manifold of a compact simple Lie group G , with $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. The normalized Ricci flow, on the space of invariant Riemannian metrics, possesses exactly two fixed points at infinity, one of them being an attracting node while the other one is a repelling one. These fixed points correspond to the two G -invariant Einstein metrics of M , namely g_1, g_2 defined above. The first metric is the (unique) Kähler-Einstein metric on M while the second one is non Kähler. Every initial metric belonging to region I and II or the line γ_2 (see Figure (2)) tends to the g_2 metric, while the metrics belonging to the line γ_1 tend to the Kähler-Einstein metric, under the normalized Ricci flow.*

4.2. Dynamics of the normalized Ricci flow on flag manifolds $M = G/K$ of Type I. In this section we study the global behavior of the normalized Ricci flow equation for a flag manifold $M = G/K$ with three isotropy summands and Type I, namely the system (16). This system is not polynomial, but

after multiplication with the (positive, in the first octant) factor $2d_1d_2d_3(d_1+d_2+d_3)(d_1+4d_2+9d_3)x_1^2x_2x_3$ it reduces to:

$$\begin{aligned}
\dot{x}_1 &= d_2x_1 \left(2d_3x_1((d_1+d_2)(d_2+d_3)x_1^2 + d_1(d_1+4d_2+9d_3)x_1x_2 - (d_1+d_2)(2d_1+d_2+d_3)x_2^2) \right. \\
&\quad + (-4d_1(-2d_2^2+d_1d_3-5d_2d_3)x_1^2 + 2d_1(2d_1+d_2+d_3)(d_1+4d_2+9d_3)x_1x_2 \\
&\quad \left. - (3d_1+2(d_2+d_3))(-d_2d_3+d_1(d_2+2d_3))x_2^2)x_3 - 2(d_1+d_2)d_3(2d_1+d_2+d_3)x_1x_3^2 \right), \\
\dot{x}_2 &= d_1x_2 \left(2d_3x_1(- (d_1+d_2)(d_1+2d_2+d_3)x_1^2 + d_2(d_1+4d_2+9d_3)x_1x_2 + (d_1+d_2)(d_1+d_3)x_2^2) \right. \\
&\quad + (-4(d_1+2d_2+d_3)(-2d_2^2+d_1d_3-5d_2d_3)x_1^2 + 2d_1d_2(d_1+4d_2+9d_3)x_1x_2 \\
&\quad \left. + (d_1+d_3)(-d_2d_3+d_1(d_2+2d_3))x_2^2)x_3 - 2(d_1+d_2)d_3(d_1+2d_2+d_3)x_1x_3^2 \right), \\
\dot{x}_3 &= d_1d_2x_3 \left(-2(d_1+d_2+2d_3)x_1((d_1+d_2)x_1^2 - (d_1+4d_2+9d_3)x_1x_2 + (d_1+d_2)x_2^2) \right. \\
&\quad + (4(2d_2^2-d_1d_3+5d_2d_3)x_1^2 + 2d_1(d_1+4d_2+9d_3)x_1x_2 \\
&\quad \left. + (d_2d_3-d_1(d_2+2d_3))x_2^2)x_3 + 2(d_1+d_2)^2x_1x_3^2 \right). \tag{19}
\end{aligned}$$

The analysis of a 3-dimensional dynamical system, like the one above, is of course a quite challenging task, but since we are interested here only in the most elementary of its properties we may proceed as follows.

We once again remark that we study this system in the region $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 / x_1, x_2, x_3 > 0\}$, since we are interested in Riemannian metrics. In this region, and in analogy with Lemma (1), we have the following:

Lemma 2. *The coordinate planes, along with the straight line $\rho(t) = (t, 2t, 3t)$, remain invariant under the flow the system (19) defines. Moreover, this system possesses no fixed points in the region P , for any value of the parameters (d_1, d_2, d_3) reported in Table 4.*

Translated into the language of Ricci flow, the above lemma ensures that the normalized Ricci flow possesses no singularities in finite region. Moreover, the invariance of the coordinate planes prohibits the evolution of a degenerate, or semi-Riemannian metric to a Riemannian one, while the invariant straight line is once again related with the existence of an Einstein metric we are about to obtain.

Since there no singularities in the finite region, we now turn our attention to the study if system (19) at infinity. We apply the Poincaré compactification procedure previously described and arrive to the following system, written in the (U_1, ϕ_1) chart:

$$\begin{aligned}
\dot{z}_1 &= (d_1+d_2+d_3)z_1 \left(2d_2^2d_3(-1-z_1^2(-1+z_2)+z_2^2) + d_1^2(d_2(-2+z_1)z_1z_2 + \right. \\
&\quad \left. 2d_3(-1+z_1^2-z_2)(1+z_2)) + d_1d_2(2d_2(-2+z_1)^2z_2 + d_3(-4+20z_2+z_1(4z_1+3(-6+z_1)z_2))) \right), \\
\dot{z}_2 &= -2d_2(d_1+d_2+d_3)z_2 \left(d_2d_3(-1+z_1^2-z_2)(-1+z_2) + d_1^2(1+z_1^2+z_1(-1+z_2)-z_2^2) \right. \\
&\quad \left. + d_1(d_3-d_2(-1+z_2)(1+(-4+z_1)z_1+z_2) - d_3(z_1(9+z_1)+z_1(-9+2z_1)z_2+z_2^2)) \right), \\
\dot{z}_3 &= -d_2 \left(2d_3((d_1+d_2)(d_2+d_3) + d_1(d_1+4d_2+9d_3)z_1 - (d_1+d_2)(2d_1+d_2+d_3)z_1^2) \right. \\
&\quad + (4d_1(2d_2^2-d_1d_3+5d_2d_3) + 2d_1(2d_1+d_2+d_3)(d_1+4d_2+9d_3)z_1 - \\
&\quad \left. (3d_1+2(d_2+d_3))(-d_2d_3+d_1(d_2+2d_3))z_1^2)z_2 - 2(d_1+d_2)d_3(2d_1+d_2+d_3)z_2^2 \right)z_3.
\end{aligned}$$

The behavior at infinity is governed by the system obtained after setting $z_3 = 0$ to the previous equations. It reads as:

$$\begin{aligned}
\dot{z}_1 &= (d_1+d_2+d_3)z_1 \left(2(d_1+d_2)^2d_3(-1+z_1^2) + (8d_1d_2^2-4d_1(d_1-5d_2)d_3-2d_1d_2(d_1+4d_2+9d_3)z_1 \right. \\
&\quad \left. + (d_1+2d_2)(-d_2d_3+d_1(d_2+2d_3))z_1^2)z_2 + 2(-d_1^2+d_2^2)d_3z_2^2 \right), \\
\dot{z}_2 &= -2d_2(d_1+d_2+d_3)z_2 \left(d_2d_3(-1+z_1^2-z_2)(-1+z_2) + d_1^2(1+z_1^2+z_1(-1+z_2)-z_2^2) \right. \\
&\quad \left. + d_1(d_3-d_2(-1+z_2)(1+(-4+z_1)z_1+z_2) - d_3(z_1(9+z_1)+z_1(-9+2z_1)z_2+z_2^2)) \right). \tag{20}
\end{aligned}$$

It is not difficult to verify that the system above possesses always a singularity, located at $(2, 3)$, which is a repelling node. Since it is complex enough to prevail the analytical study of other fixed points, we substitute the values of the dimensions d_1, d_2, d_3 reported in Table 4, and study each case separately, given the corresponding form of system (20). In any case we compute two more fixed points which are saddles.

- $G_2/U(2)$

$$\begin{aligned}\dot{z}_1 &= 10z_1(288(-1 + z_1^2) + (512 - 768z_1 + 256z_1^2)z_2 - 96z_2^2) \\ \dot{z}_2 &= 640z_2(-3 + 12z_1 + 2(-6 + z_1)z_1z_2 + 3z_2^2)\end{aligned}$$

Equilibria: $(2, 3)$, $(0.186894, 0.981478)$ and $(1.67467, 2.05238)$.

- $E_6/SU(3) \times SU(3) \times SU(2) \times U(1)$

$$\begin{aligned}\dot{z}_1 &= 58z_1(23328(-1 + z_1^2) + (124416 - 186624z_1 + 62208z_1^2)z_2 - 7776z_2^2) \\ \dot{z}_2 &= 902016z_2(-5 - 4(-3 + z_1)z_1 + 2(-6 + z_1)z_1z_2 + 5z_2^2)\end{aligned}$$

Equilibria: $(2, 3)$, $(0.771752, 1.33186)$ and $(1.04268, 0.373467)$.

- $E_7/SU(5) \times SU(3) \times U(1)$

$$\begin{aligned}\dot{z}_1 &= 4233600z_1(3(-1 + z_1^2) + 7(2 + (-3 + z_1)z_1)z_2 - z_2^2) \\ \dot{z}_2 &= 2116800z_2(-17 + 42z_1 - 13z_1^2 + 7(-6 + z_1)z_1z_2 + 17z_2^2)\end{aligned}$$

Equilibria: $(2, 3)$, $(0.733552, 1.27681)$ and $(1.06029, 0.443559)$.

- $E_7/SU(6) \times SU(2) \times U(1)$

$$\begin{aligned}\dot{z}_1 &= 2030400z_1(3(-1 + z_1^2) + 12(2 + (-3 + z_1)z_1)z_2 - z_2^2) \\ \dot{z}_2 &= 4060800z_2(-8 + 18z_1 - 7z_1^2 + 3(-6 + z_1)z_1z_2 + 8z_2^2)\end{aligned}$$

Equilibria: $(2, 3)$, $(0.85368, 1.45259)$ and $(1.01573, 0.229231)$.

- $E_8/E_6 \times SU(2) \times U(1)$

$$\begin{aligned}\dot{z}_1 &= 11617344z_1(3(-1 + z_1^2) + 20(2 + (-3 + z_1)z_1)z_2 - z_2^2) \\ \dot{z}_2 &= 23234688z_2(-14 + 30z_1 - 13z_1^2 + 5(-6 + z_1)z_1z_2 + 14z_2^2)\end{aligned}$$

Equilibria: $(2, 3)$, $(0.914286, 1.54198)$ and $(1.0049, 0.129681)$.

- $E_8/SU(8) \times U(1)$

$$\begin{aligned}\dot{z}_1 &= 18464768z_1(9(-1 + z_1^2) + 20(2 + (-3 + z_1)z_1)z_2 - 3z_2^2) \\ \dot{z}_2 &= 36929536z_2(-30z_1(-1 + z_2) + z_1^2(-9 + 5z_2) + 12(-1 + z_2^2))\end{aligned}$$

Equilibria: $(2, 3)$, $(0.177586, 1.25432)$ and $(1.06853, 0.473177)$.

- $F_4/SU(3) \times SU(2) \times U(1)$

$$\begin{aligned}\dot{z}_1 &= 138240z_1(3(-1 + z_1^2) + 6(2 + (-3 + z_1)z_1)z_2 - z_2^2) \\ \dot{z}_2 &= 138240z_2(-7 + 18z_1 - 5z_1^2 + 3(-6 + z_1)z_1z_2 + 7z_2^2)\end{aligned}$$

Equilibria: $(2, 3)$, $(0.678535, 1.20122)$ and $(1.09057, 0.546045)$.

Since the (U_1, ϕ_1) chart corresponds to the $y_1 = 1$ plane, we consider the metrics whose coefficients are given by $(1, a, b)$, where a, b are equal to the coordinates of the fixed points just calculated. These metrics are invariant Einstein metrics and the one with coefficients $(1, 2, 3)$ is the unique Kähler–Einstein which admits M . We have thus proved the following:

Theorem 4. *Let $M = G/K$ be a generalized flag manifold of a compact simple Lie group G of Type I. The normalized Ricci flow, on the space of invariant Riemannian metrics on M , possesses exactly three singularities at infinity. The point $(2, 3)$ is a repelling node while the other two are saddle points. These fixed points correspond to the three (up to scale) G -invariant Einstein metrics which admits M .*

4.3. Conclusions. We studied in this paper the behavior of the normalized Ricci flow on flag manifolds with two or three isotropy summands. The Ricci flow equation reduces to a system of two, correspondingly three, ordinary differential equations.

In the case of two isotropy summands we were able to completely determine the system's global phase portrait (at the first quadrant of the plane, since we are interested in Riemmanian metrics), using standard techniques of Dynamical Systems theory. In the case of three isotropy summands the problem becomes (as usual) more complicated, making the presentation of the corresponding complete phase portrait impossible. In any case, we were able to calculate explicitly the invariant Einstein metrics that exist, as the fixed points of the normalized Ricci flow at infinity. Moreover, we can point out the Kähler–Einstein metric as the fixed point having no stable eigendirections.

Obtaining the invariant Einstein metrics using the Ricci flow does not only provides their explicit number and form, but also eliminates the asymptotic behavior of G -invariant Riemmanian metrics. This unified method may prove to be useful in obtaining more general results about the existence and the classification problem of homogeneous Einstein metrics on compact homogeneous spaces.

REFERENCES

- [1] D. V. Alekseevsky and A. Arvanitoyeorgos: *Riemannian flag manifolds with homogeneous geodesics*, Trans. Amer. Math. Soc. 359 (8) (2007) 3769–3789.
- [2] D. V. Alekseevsky and A. M. Perelomov: *Invariant Kähler-Einstein metrics on compact homogeneous spaces*, Funct. Anal. Appl. 20 (3) (1986) 171–182.
- [3] A. Arvanitoyeorgos: *An Introduction to Lie Groups and the Geometry of Homogeneous Spaces*, Amer. Math. Soc, Student Math. Library, Vol. 22, 2003.
- [4] A. Arvanitoyeorgos and I. Chrysikos: *Motion of charged particles and homogeneous geodesics in Kähler C -spaces with two isotropy summands*, Tokyo J. Math. (32) 2 (2009) 1-14.
- [5] A. Arvanitoyeorgos and I. Chrysikos: *Invariant Einstein metrics on generalized flag manifolds with two isotropy summands*, to appear in J. Aust. Math. Soc. (2010).
- [6] A. Arvanitoyeorgos and I. Chrysikos: *Invariant Einstein metrics on generalized flag manifolds with four isotropy summands*, Ann. Glob. Anal. Geom. 37 (2010) 185-219.
- [7] A. L. Besse: *Einstein Manifolds*, Springer-Verlag, Berlin, 1986.
- [8] C. Böhm and B. Wilking: *Nonnegatively curved manifolds with finite fundamental groups admits metrics with positive Ricci curvature*, Geom. Funct. Anal. 17 (2007) 665-681.
- [9] A. Borel and F. Hirzebruch: *Characteristic classes and homogeneous spaces I*, Amer. J. Math. 80 (1958) 458–538.
- [10] H-D. Chao and B. Chow: *Recent developments on the Ricci flow*, Bull. Amer. Soc. 36 (1999) 59-74.
- [11] R. Dohira: *Geodesics in reductive homogeneous spaces*, Tsukuba J. Math. 19 (1) (1995) 233-243.
- [12] M. Kimura: *Homogeneous Einstein metrics on certain Kähler C -spaces*, Adv. Stud. Pure Math. 18-I (1990) 303–320.
- [13] W. Dickinson and M. Kerr: *The geometry of compact homogeneous spaces with two isotropy summands*, Ann. Glob. Anal. Geom. 34 (2008) 329-350.
- [14] D. Glickenstein and T. L. Payne: *Ricci flow on three-dimensional, unimodular metric Lie algebras*, preprint (2009), arXiv:0909.0938v1.
- [15] E.A.V. Gonzales: *Generic properties of polynomial vector fields at infinity*, Trans. Amer. Math. Soc. 143:201-222, 1969.
- [16] L. Grama and R. M. Martins: *The Ricci flow of left-invariant metrics on full flag manifold $SU(3)/T$ from a dynamical systems point of view*, Bull. Sci. Math. 133 (2009) 463-469. ,
- [17] S. Helgason: *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York 1978
- [18] R. S. Hamilton: *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982) 255-306.
- [19] R. S. Hamilton: *Four-manifolds with positive curvature operator*, J. Differential Geom. 24 (1986) 153-179.
- [20] J-S. Park and Y. Sakane: *Invariant Einstein metrics on certain homogeneous spaces*, Tokyo J. Math. 20 (1) (1997) 51–61.
- [21] H. Poincaré: *Sur l' integration des équations différentielles du premier ordre et du premier degré I*, Rendiconti del circolo matematico di Palermo, 5, (1891) 161-191.
- [22] Y. Sakane *Homogeneous Einstein metrics on principal circle bundles II*, in Differential Geometry, Proceedings of the symposium in honour of Professor Su Buchin on his 90th birthday (Editors: C. H. Gu, H. S. Hu, Y. L. Xin), World Scientific Publishing, (1993) 177–186.

- [23] M. Wang and W. Ziller: *Existence and non-existence of homogeneous Einstein metrics*, Invent. Math. 84 (1986) 177–194.

CENTER OF RESEARCH AND APPLICATIONS OF NONLINEAR SYSTEMS (CRANS), UNIVERSITY OF PATRAS,
DEPARTMENT OF MATHEMATICS, GR-26500 RION, GREECE

E-mail address: `SAnastassiou@gmail.com`

UNIVERSITY OF PATRAS, DEPARTMENT OF MATHEMATICS, GR-26500 RION, GREECE

E-mail address: `xrysikos@master.math.upatras.gr`